NORMAL SUBGROUPS

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Post date: 14 January 2026.

Reference: *A Gentle Introduction to Group Theory*, Bana Al Subaiei & Muneerah Al Nuwairan, Section 7.5.

A *normal subgroup* H of a group G, written $H \trianglelefteq G$ is a subgroup that satisfies the following conditions. For all elements $a \in G$, we have

(1) $aH = Ha$. That is, the left and right cosets of H in G are equal.
(2) $aHa^{-1} = H$.
(3) $aHa^{-1} \subseteq H$.
(4) $aH \subseteq Ha$.

Actually, the condition 4 above is automatically true if $aH = Ha$, but sometimes it requires less work to demonstrate condition 4 than the strict equality. Condition 3 is also automatically true if condition 2 is true, but again, it may require less work to show condition 3. Proofs of these conditions are given as Proposition 7.5.2 in the above reference.

Example 1. Find the normal subgroups of the symmetric group $\mathfrak{S}_3$, the permutation group on 3 objects. First, we can list all the subgroups of $\mathfrak{S}_3$:

$$S_1 = \{e\} \tag{1}$$
$$S_2 = \{e, (12)\} \tag{2}$$
$$S_3 = \{e, (13)\} \tag{3}$$
$$S_4 = \{e, (23)\} \tag{4}$$
$$S_5 = \{e, (123), (132)\} \tag{5}$$
$$S_6 = \{e, (12), (13), (23), (123), (132)\} = \mathfrak{S}_3 \tag{6}$$

Only S_1, S_5 and S_6 are normal subgroups. We can see this as follows.

For S_1, it is trivially true that $aS_1 = S_1 a$ since the only element in S_1 is the identity e. For S_6 it is true that condition 3 above is true, since aS_6a^{-1} contains only elements in S_6, since $S_6 = \mathfrak{S}_3$, the full symmetric group.

To show that S_5 is normal, we can apply condition 3. This is a bit tedious but we can run through the calculations. (Remember that when combining permutations we work from right to left.) For the identity e, we have

$$aea^{-1} = aa^{-1} = e \in S_5 \tag{7}$$

Letting a be each of (12), (13) and (23) in turn, we see that in all 3 cases $a=a^{-1}$. We have

$$(12)(123)(12)=(132)\in S_5 \tag{8}$$
$$(13)(123)(13)=(132)\in S_5 \tag{9}$$
$$(23)(123)(23)=(132)\in S_5 \tag{10}$$
$$(12)(132)(12)=(123)\in S_5 \tag{11}$$
$$(13)(132)(13)=(123)\in S_5 \tag{12}$$
$$(23)(132)(23)=(123)\in S_5 \tag{13}$$

Next, we let a be (123), and thus $a^{-1}=(132)$. We have

$$(123)(123)(132)=(123)\in S_5 \tag{14}$$
$$(123)(132)(132)=(132)\in S_5 \tag{15}$$

Finally, we let a be (132), with $a^{-1}=(123)$. We have

$$(132)(123)(123)=(123)\in S_5 \tag{16}$$
$$(132)(132)(123)=(132)\in S_5 \tag{17}$$

To show that S_2, $S_3$and S_4 are *not* normal subgroups, we can show that each one violates condition 3. For example, for S_2, we have

$$(23)(12)(23)=(13)\notin S_2 \tag{18}$$

Similarly, we have for S_3

$$(23)(13)(23)=(12)\notin S_3 \tag{19}$$

and for S_4

$$(12)(23)(12)=(13)\notin S_4 \tag{20}$$

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